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NSR superstring measures revisited

A. Morozov

ITEP,
B.Cheremushkinskaya, 25, Moscow, Russia
E-mail: morozov@itep.ru

Abstract: Review of remarkable progress in evaluation of NSR superstring measures, originated by E.D'Hoker and D.Phong. These recent results are presented in the old-fashioned form, what allows to highlight the options which have been overlooked in original considerations in late 1980's.

Keywords: Superstrings and Heterotic Strings, Bosonic Strings.

Contents

1. Introduction

After the role of holomorphicity in $2d$ conformal theories was fully realized and exploited in [1] it was natural to look for the holomorphic factorization in the conformal-invariant first-quantized theories of critical strings [2]. The problem here was that the relevant quantities had to be meromorphic not only in z-variables, which define positions of operators in operator-product expansions, but also in the moduli of Riemann surfaces. The relevant holomorphic anomalies in Polyakov's combination of determinants, which define string measures for bosonic, super- and heterotic strings, were evaluated in [3] and shown to vanish together with conformal anomaly of [2]. This Belavin-Knizhnik theorem became a starting point for construction of perturbative string and conformal field theories, reviewed, for example, in [4]–[8]. Without Belavin-Knizhnik theorem the Polyakov string measures could be discussed in terms of either Shottky parametrization [9] or Selberg traces [10]. With this theorem the adequate language became that of the Mumford measure $d\mu$ on the moduli space of complex curves (= Riemann surfaces) [11, 12]: the measure for bosonic string was proved in [3] to be $\frac{|d\mu|^2}{\det(\text{Im } T)^{13}}$, while that for the NSR superstring [13] had to contain an extra factor of $(\det\ (\text{Im } T))^8$ with $d\mu$ presumably multiplied by some modular form of the weight 8. The first big success on this way was explicit construction of $d\mu$ for the genera $2, 3$ and 4 in terms of period matrices in [14 – 16] — and this was the starting point of the long road towards DHP construction of NSR measures in [17]–[37].

From the very beginning there were two related but different strategies.

The first approach was to begin with Polyakov's measure for NSR string at given characteristic e, expressed through determinants in [2] and holomorphically factorized in [3], integrate away the "supermoduli" and obtain the relevant modification $d\mu[e]$ of the Mumford measure. This road looked straightforward [38]–[52], until it was shown in [53 – 55] that naive integration over supermoduli does not work and its proper version requires a lot of work. This work was finally done by Eric D'Hoker and Duong Phong (DHP) in a series of impressive papers [17]–[25], but only 15 year later and only for genus 2 so far.

The second equally obvious approach was to make educated guesses for NSR superstring measure, i.e. to find the relevant weight-8 modular forms from their expected properties, at least for the first low genera, like it was done in [14 – 16] for $d\mu$ itself. As explained in [41], the main obstacle on this way was modular non-invariance of the Riemann identities – which are necessarily used for cancelation of tachionic divergencies after GSO projection (=sum over characteristics) [56]. After a series of attempts [57] — now known to be partly misleading — this approach was temporarily abandoned. Now, after the DHP triumph it is used again and already led to explicit construction of NSR measures at genera 3 [34], 4 [36] and – somewhat less explicitly — for all higher genera [35]. The problem for $g > 4$ is that the Mumford measure $d\mu$ does not possess any nice representation in terms modular forms (only a far more transcendental formulas of [6, 53, 8] are currently available), but the result of [35] supports the original suggestion of [3, 15] that the ratio $\Xi_8[e] = d\mu[e]/d\mu$ is a modular form (then it has modular weight 8) and this $\Xi_8[e]$ is proposed in [35] in a simple and clear form. The only remaining problem with these suggestions at $g \geq 3$ is related to $1, 2, 3, 4$-functions, and this makes the story of NSR measures not fully completed. Still, we already know quite a lot, and the time probably came to analyze and explain the failures of the early attempts and understand what are the answers to the questions, posed but unanswered in late 1980's. This paper is an attempt of such analysis.

2. Riemann surfaces and theta constants [58]–[63]

2.1 Theta-functions, theta-constants and modular forms on the Siegel semi-space

2.1.1 Theta functions

Theta-functions are special functions, associated with abelian varieties: g-dimensional tori,

g	N_e	N_*
1	3	1
2	10	6
3	36	28
4	136	120
...		

Table 1: The number of even (N_e) and odd (N_*) θ-characteristics for low genera g.

which are factors of C^g over relations $z_i \sim z_i + T_{ij}z_j$, where *symmetric* period matrices T_{ij} with positive definite imaginary part (Im T) are points in the $g(g+1)/2$ Siegel semi-space, defined modulo integer symplectic (also called *modular*) transformations $T \sim (AT+B)/(CT+D)$ from the group $\mathrm{Sp}(g,Z)$.

Bosonic and super-string measures on the moduli space of Riemann surfaces are defined in terms of theta-functions with semi-integer characteristics, this is taken into account in the following definition:

$$\theta \begin{bmatrix} \vec{\delta} \\ \vec{\varepsilon} \end{bmatrix} (\vec{z}|T) = \sum_{\vec{n}\in Z^g} \exp\left\{ i\pi \left(\vec{n}+\frac{1}{2}\vec{\delta}\right) T \left(\vec{n}+\frac{1}{2}\vec{\delta}\right) + 2\pi i \left(\vec{n}+\frac{1}{2}\vec{\delta}\right)\left(\vec{z}+\frac{1}{2}\vec{\varepsilon}\right)\right\} \tag{2.1}$$

Sums are over all g vectors $\vec{n}$ with integer coordinates, each coordinate of characteristic vectors $\vec{\delta}$ and $\vec{\varepsilon}$ can take values 0 or 1. Characteristic is called even or odd if scalar product $\vec{\delta}\vec{\varepsilon}$ is even or odd respectively and associated theta-function is even or odd in $\vec{z}$. The value of theta-function at $\vec{z}=0$ is called theta-constant, it automatically vanishes for odd characteristic. We often denote characteristics by $e = \{\vec{\delta},\vec{\varepsilon}\}$, in most cases these will be even characteristics, when we refer to *some* odd characteristic it is labeled by $*$. There are $N_e = 2^{g-1}(2^g+1)$ even and $N_* = 2^{g-1}(2^g-1)$ odd semi-integer characteristics, see table 1. With a pair of characteristics (not obligatory even) we associate a sign factor

$$\langle e_1, e_2\rangle \ = \exp\left\{ i\pi\left(\vec{\delta}_1\vec{\varepsilon}_2 - \vec{\varepsilon}_1\vec{\delta}_2\right)\right\} = \left(\vec{\delta}_1\vec{\varepsilon}_2 - \vec{\varepsilon}_1\vec{\delta}_2\right) \bmod 2 = \ \langle e_2, e_1\rangle \tag{2.2}$$

which takes values ± 1. In particular, $\langle e, e\rangle \ = 1$.

2.1.2 Modular forms

Functions of T, transforming multiplicatively under modular transformations, $f(T) \to (\det(CT+D))^{-k} f(T)$, are called modular forms of weight k. Theta-constants are not modular forms, they are not simply multiplied by $(\det(CT+D))^{-1/2}$, but also acquire additional numerical factors proportional to $e^{i\pi/4}$ and change characteristics.

The simplest modular forms can be made from the 8-th powers of θ-constants, since modular transformations act on them just by permuting their characteristics. In particular, for any integer k and g

$$\xi_{4k} \equiv \sum_e^{N_e} \theta_e^{8k} \tag{2.3}$$

is a modular form of weight $4k$. Important for NSR measures are

$$\xi_4 = \sum_e^{N_e} \theta_e^8 \quad \text{and} \quad \xi_8 = \sum_e^{N_e} \theta_e^{16} \tag{2.4}$$

Also

$$\Pi \equiv \prod_e^{N_e} \theta_e \tag{2.5}$$

of weight $N_e/2$ is a modular form for $g \geq 3$, while roots of unity arise and Π should be raised to power 8 and 2 at $g = 1$ and $g = 2$ respectively. This Π is the building block of Mumford measure at $g = 1, 2, 3$, see section 3 below.

However, the set of modular forms is by no means exhausted by these trivial characters of the permutation group. Most important are other examples, having the same form for all g, like

$$\xi_{2+4k,2+4l} \equiv \sum_{e,e'}^{N_e} \langle e, e' \rangle \theta_e^{4+8k} \theta_{e'}^{4+8l} = \sum_e^{N_e} \theta_e^{4+8k} \xi_{2+4l}[e] \tag{2.6}$$

which has weight $4(k + l + 1)$. Modular invariance of $\xi_{2+4k,2+4l}$ implies that

$$\xi_{2+4l}[e] \equiv \sum_{e'}^{N_e} \langle e, e' \rangle \theta_{e'}^{4+8l} \tag{2.7}$$

transforms under modular transformations exactly like ξ_e^4 (we call such forms "semi-modular"). The sign factors $\langle e, e' \rangle$ serve to restore modular invariance whenever $\theta_{e'}^4$ appear instead of $\theta_{e'}^8$.

As discovered in [17]–[25],[33, 34, 36] and formulated in a very clear and general form in [35], superstring measures are actually constructed from a wider family of modular forms of weight 8, of which ξ_8, and ξ_4^2 and $\xi_{2,6}$ are just the first three members:

$$\xi_8^{(p)} = \sum_e^{N_e} \xi_8^{(p)}[e] \tag{2.8}$$

where

$$\begin{aligned}
\xi_8^{(0)}[e] &= \theta_e^{16}, && \text{i.e.} && \xi_8^{(0)} = \xi_8, \\
\xi_8^{(1)}[e] &= \theta_e^8 \sum_{e_1}^{N_e} \theta_{e+e_1}^8 = \theta_e^8 \xi_4, && \text{i.e.} && \xi_8^{(1)} = \xi_4^2, \\
\xi_8^{(2)}[e] &= \theta_e^4 \sum_{e_1,e_2}^{N_e} \theta_{e+e_1}^4 \theta_{e+e_2}^4 \theta_{e+e_1+e_2}^4, \\
\xi_8^{(3)}[e] &= \theta_e^2 \sum_{e_1,e_2,e_3}^{N_e} \theta_{e+e_1}^2 \theta_{e+e_2}^2 \theta_{e+e_3}^2 \theta_{e+e_1+e_2}^2 \theta_{e+e_1+e_3}^2 \theta_{e+e_2+e_3}^2 \theta_{e+e_1+e_2+e_3}^2, \\
&\dots
\end{aligned} \tag{2.9}$$

and in general

$$\xi_8^{(p)}[e]=\sum_{e_1,\dots,e_p}^{N_e}\left\{\theta_e\cdot\left(\prod_i^p\theta_{e+e_i}\right)\cdot\left(\prod_{i<j}^p\theta_{e+e_i+e_j}\right)\cdot\left(\prod_{i<j<k}^p\theta_{e+e_i+e_j+e_k}\right)\cdot\ldots\cdot\theta_{e+e_1+\dots+e_p}\right\}^{4/2^p} \tag{2.10}$$

Characteristics are added as vectors. Sign factors $\langle e,e'\rangle$ are not seen in these formulas, because, say, in $\xi_8^{(2)}$

$$\langle e,e+e_1\rangle\langle e,e+e_2\rangle\langle e,e+e_1+e_2\rangle \;=\; \langle e,e_1\rangle^2\langle e,e_2\rangle^2=1,$$

while in $\xi_8^{(3)}$

$$\begin{aligned}&\sqrt{\langle e,e+e_1\rangle}\sqrt{\langle e,e+e_2\rangle}\sqrt{\langle e,e+e_3\rangle}\sqrt{\langle e,e+e_1+e_2\rangle}\sqrt{\langle e,e+e_1+e_3\rangle}\sqrt{\langle e,e+e_2+e_3\rangle}\cdot\\ &\qquad\cdot\sqrt{\langle e,e+e_1+e_2+e_3\rangle}=\sqrt{\langle e,e_1\rangle^4\langle e,e_2\rangle^4\langle e,e_3\rangle^4}=\langle e,e_1\rangle^2\langle e,e_2\rangle^2\langle e,e_3\rangle^2=1\end{aligned}$$

and so on. Many terms in the sums (2.9) and (2.10) are actually vanishing, because contributing characteristics are odd, for careful analysis of this phenomenon in terms of isotropic spaces and Lagrange varieties see [34]. Only $\xi_8^{(p)}$ with $p\le g$ appear in NSR measures in section 4.2 below. For $g\ge 5$ fractional powers of theta-constants begin to appear in the relevant $\xi_8^{(p)}$, see [37] for an (optimistic) analysis of the $g=5$ case.

2.1.3 Grushevsky's basis

In [35] a slightly different basis was actually used, with all diagonal terms eliminated from the sums (2.9) and (2.10):

$$\begin{aligned}
\xi_8^{(0)}[e] &= G_8^{(0)}[e],\\
\xi_8^{(1)}[e] &= G_8^{(0)}[e]+G_8^{(1)}[e],\\
\xi_8^{(2)}[e] &= G_8^{(0)}[e]+3G_8^{(1)}[e]+G_8^{(2)}[e],\\
\xi_8^{(3)}[e] &= G_8^{(0)}[e]+7G_8^{(1)}[e]+7G_8^{(2)}[e]+G_8^{(3)}[e],\\
\xi_8^{(4)}[e] &= G_8^{(0)}[e]+15G_8^{(1)}[e]+35G_8^{(2)}[e]+15G_8^{(3)}[e]+G_8^{(4)}[e],\\
\xi_8^{(5)}[e] &= G_8^{(0)}[e]+31G_8^{(1)}[e]+155G_8^{(2)}[e]+155G_8^{(3)}[e]+31G_8^{(4)}[e]+G_8^{(5)}[e],\\
\xi_8^{(6)}[e] &= G_8^{(0)}[e]+63G_8^{(1)}[e]+651G_8^{(2)}[e]+1395G_8^{(3)}[e]+651G_8^{(4)}[e]+63G_8^{(5)}[e]+G_8^{(6)}[e],\\
&\dots
\end{aligned} \tag{2.11}$$

and in general

$$\begin{aligned}\xi_8^{(p)}[e] = {}& G_8^{(p)}[e]+(2^p-1)G_8^{(p-1)}[e]+\frac{(2^p-1)(2^{p-1}-1)}{3}G_8^{(p-2)}[e]\\
&+\frac{(2^p-1)(2^{p-1}-1)(2^{p-2}-1)}{7\cdot 3}G_8^{(p-3)}[e]\\
&+\frac{(2^p-1)(2^{p-1}-1)(2^{p-2}-1)(2^{p-3}-1)}{15\cdot 7\cdot 3}G_8^{(p-4)}[e]\\
&+\frac{(2^p-1)(2^{p-1}-1)(2^{p-2}-1)(2^{p-3}-1)(2^{p-4}-1)}{31\cdot 15\cdot 7\cdot 3}G_8^{(p-5)}[e]+\cdots\end{aligned}$$

(note the reversed order of terms in the last formula). The definition of, say, $G_8^{(1)}$ is

$$G_8^{(1)}[e] \equiv \theta_e^8 \sum_{e_1\neq 0}^{N_e} \theta_{e+e_1}^8 = \theta_e^8 \left(\sum_{e_1}^{N_e} \theta_{e+e_1}^8 - \theta_e^8 \right) = \xi_8^{(1)}[e] - \xi_8^{(0)}[e] \tag{2.12}$$

In other words, in the sum for $\xi_8^{(1)}[e]$ there is one term with $e_1 = 0$, which is $G_8^{(0)}$, and all the rest is $G_8^{(1)}$. Similarly, in the double sum for $\xi_8^{(2)}$ there is a contribution from $e_1 = e_2 = 0$ — this is $G_8^{(0)}$,– there are contributions from either $e_1 = 0$ and $e_2 \neq 0$ or $e_2 = 0$ and $e_1 \neq 0$ or $e_1 + e_2 = 0$ and $e_1 = e_2 \neq 0$ – these are $3 \cdot G_8^{(1)}$,– and the rest is $G_8^{(2)}$. When we proceed to triple sums, it is important to remember that $e_1 = e_2 = 0$ automatically implies that $e_1 + e_2 = 0$: this will produce factors like $2^p - 4 = 4(2^{p-2} - 1)$ instead of $2^p - 3$ when we select the third characteristic to nullify after the two are already chosen.

There is no a priori reason to prefer $G_8^{(p)}$ over $\xi_8^{(p)}$, but in [35] it was demonstrated that NSR measures are actually "more universal" (coefficients do not depend on g) when expressed in terms of $G_8^{(p)}$, see section 4.2 below.

2.1.4 Riemann identities

There are no non-vanishing modular forms of weight 2 made from the 4-th powers of theta-constants, instead there is a set of Riemann identities

$$\mathcal{R}_* \equiv \sum_e^{N_e} \langle e, *\rangle \theta_e^4 = 0 \tag{2.13}$$

for all of the N_* odd characteristics $*$. Of $N_* = 2^{g-1}(2^g - 1)$ Riemann identities there are $\frac{1}{3}(4^g - 1) = \frac{1}{3}(2^g + 1)(2^g - 1)$ linearly independent, and they reduce the number of *linearly*-independent $\theta^4[e]$ from $N_e = 2^{g-1}(2^g + 1)$ to $\frac{1}{3}(2^g - 1)(2^g + 1)$. Other relations between theta-constants involve powers of θ^4. In naive superstring considerations an even stronger version of Riemann identity is commonly used, where up to three of the four theta-constants are promoted to theta-functions:

$$\mathcal{R}_*(\vec{z}_1, \vec{z}_2, \vec{z}_3|T) \equiv \sum_e^{N_e} \langle e, *\rangle \theta_e(\vec{0})\theta_e(\vec{z}_{12})\theta_e(\vec{z}_{23})\theta_e(\vec{z}_{31}) = 0 \tag{2.14}$$

for any three vectors $\vec{z}_1$, $\vec{z}_2$, $\vec{z}_3$. Both (2.13) and (2.14) are corollaries of a general relation

$$\begin{aligned} &\sum_{\text{all } e} \langle e, *\rangle \theta_e(\vec{z}_1)\theta_e(\vec{z}_2)\theta_e(\vec{z}_3)\theta_e(\vec{z}_4) \\ &= 2^g \theta_*\left(\frac{\vec{z}_1+\vec{z}_2+\vec{z}_3+\vec{z}_4}{2}\right)\theta_*\left(\frac{\vec{z}_1+\vec{z}_2-\vec{z}_3-\vec{z}_4}{2}\right)\theta_*\left(\frac{\vec{z}_1-\vec{z}_2+\vec{z}_3-\vec{z}_4}{2}\right)\theta_*\left(\frac{\vec{z}_1-\vec{z}_2-\vec{z}_3+\vec{z}_4}{2}\right) \end{aligned} \tag{2.15}$$

If one needs a sum over even characteristics at the l.h.s. it is enough to add the same formula with $\vec{z}_4 \to -\vec{z}_4$ to the r.h.s. (and divide by two). In particular,

$$\sum_e \langle e, *\rangle \theta_e(\vec{0})^3 \theta_e(\vec{z}) = 2^g \theta_*^4\left(\frac{\vec{z}}{2}\right), \tag{2.16}$$

plays important role in superstring calculus.

2.1.5 Decomposition rules

For block-diagonal matrices $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with $g = g_1 + g_2$ the theta-functions factorize into products $\theta_e(\vec{z}|T) = \theta_{e_1}(\vec{z}_1|T_1)\theta_{e_2}(\vec{z}_2|T_2)$. Above-mentioned modular forms behave as multiplicative characters under this decomposition: they also factorize,

$$\begin{aligned} \xi_{4k}(T) &= \xi_{4k}(T_1)\xi_{4k}(T_2), & \xi_{2+4k,2+4l}(T) &= \xi_{2+4k,2+4l}(T_1)\xi_{2+4k,2+4l}(T_2), \\ \xi_8^{(p)}[e](T) &= \xi_8^{(p)}[e_1](T_1)\xi_8^{(p)}[e_2](T_2), & \mathcal{R}_*(T) &= \mathcal{R}_{*_1}(T_1)\mathcal{R}_{*_2}(T_2), \end{aligned} \tag{2.17}$$

while Π in (2.5) vanishes, because some even characteristics e get decomposed into two odd, for example $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2.2 Moduli space and Riemannian θ-functions [58]–[63]

Riemannian theta-functions are associated with tori which are Jacobians of Riemann surfaces (complex curves). Then g is the genus of the curve and T_{ij} is its period matrix. Period matrices define an embedding of moduli space of Riemann surfaces into Siegel semi-space, and moduli space has non-vanishing codimension $g(g+1)/2 - (3g-3)$ for $g \geq 4$. In terms of T matrices this embedding is defined by a set of transcendental *Shottky relations*. Today the best known formulation of these relations is that the corresponding theta-function is a τ-function of KP-hierarchy [64, 65] or, in other words, satisfy Wick theorem [8, 66, 67], also known as a set of Fay's identities [59]:

$$\det_{i,j} \frac{\theta_e(\vec{x}_i - \vec{y}_j)}{E(x_i, y_j)\theta_e(\vec{0})} = \frac{\theta_e(\sum_i \vec{x}_i - \sum_i \vec{y}_i)}{\theta_e(\vec{0})} \frac{\prod_{i<j} E(x_i, x_j)E(y_i, y_j)}{\prod_{i,j} E(x_i, y_j)} \tag{2.18}$$

Here $E(x, y) = \frac{\theta_*(\vec{x}-\vec{y})}{\nu_*(x)\nu_*(y)}$ is the prime form, $\vec{x} - \vec{y} = \int_y^x \vec{\omega}$ and $\nu^2(x) = \theta^*_{,i}(\vec{0})\omega_i(x)$.

Alternatively, one of the Shottky relations (the only one in the case of $g = 4$) can be formulated as the condition

$$\chi_8 \equiv\ 2^g\xi_8 - \xi_4^2 \ =\ 2^g \sum_e \theta_e^{16} - \left(\sum_e \theta_e^8\right)^2 = 0 \tag{2.19}$$

This is currently a hypothesis [3, 14, 15, 19], rigorously proved only for $g = 4$ [63] (for $g \leq 3$ this is not a Shottky relation, but a simple algebraic relation in hyperelliptic parametrization, see below). At the same time it expresses the equivalence (duality) of string compactifications on 16-dimensional tori with the two even self-dual lattices Γ_{16} and $\Gamma_8 \times \Gamma_8$ and thus of the heterotic SO(32) and $E_8 \times E_8$ strings [68] and is strongly believed to be true "on physical grounds".

2.3 Hyperelliptic surfaces [58, 59, 69]

Hyperelliptic surfaces are ramified double coverings of Riemann sphere, which can be described as

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i) \tag{2.20}$$

Hyperelliptic surfaces form a $(2g-1)$-dimensional subspace in the moduli space, parameterized by ramification points a_i modulo rational transformations $(x,y|a_i)\to\left(\frac{Ax+B}{Cx+D},\ \frac{y}{(Cx+D)^{g+1}}\middle|\frac{Aa_i+B}{Ca_i+D}\right)$. At genera 1 and 2 all Riemann surfaces are hyperelliptic. At genus 3 hyperelliptic locus has codimension 1 and is defined by $\Pi=\prod_e\theta_e=0$.

Consideration of hyperelliptic locus is very instructive, because characteristic-dependence of theta-constants on it becomes pure algebraic. Semi-integer theta-characteristics are associated with splitting of all $2g+2$ ramification points into two sets of $g+1-2k$ and $g+1+2k$ points: $\{a\}=\{\tilde{a}\}\bigcup\left\{\tilde{\tilde{a}}\right\}$. Characteristic is even/odd if k is even/odd, it is also called singular if $k>2$. Non-vanishing are only theta-constants associated with even non-singular characteristic, $k=0$, and these non-vanishing theta-constants are expressed through ramification points by Thomae formulas:

$$\theta^4[e]=\pm(\det\sigma)^2\prod_{i<j}^{g+1}(\tilde{a}_i-\tilde{a}_j)\left(\tilde{\tilde{a}}_i-\tilde{\tilde{a}}_j\right)=\pm(\det\sigma)^2\prod_{i<j}^{g+1}\tilde{a}_{ij}\tilde{\tilde{a}}_{ij} \tag{2.21}$$

Proportionality coefficient is transcendental, with $\sigma_{ij}=\oint_{A_i}\frac{x^{j-1}dx}{y(x)}$, see [58, 59, 69] for details. Fortunately, we do not need it in the present text.

In more detail Thomae formulas depend on the choice of some set U of $g+1$ ramification points. Characteristics are in one-to-one correspondence with the sets S, consisting of even numbers of ramification points. Given U and S one can define a new set $S\circ U=S\cup U-S\cap U$ and characteristic is non-singular if $\#(S\circ U)=g+1$ and in this case

$$\theta_e^4\ \sim\ (-)^{\#(S\cup U)}\prod_{\substack{\tilde{a}_i\in S\circ U\\ \tilde{\tilde{a}}_j\notin S\circ U}}\left(\tilde{a}_i-\tilde{\tilde{a}}_j\right)^{-1} \tag{2.22}$$

The sign factor for any pair of characteristics (even or odd) is

$$\langle e_1,e_2\rangle=(-)^{\#(S_1\cup S_2)} \tag{2.23}$$

The number of non-singular even characteristics is $N_{\rm nse}=C_{2g+2}^{g+1}$, so that $N_{\rm nse}=N_e$ for $g=1,2$, while $N_{\rm nse}=N_e-1$ for $g=3$ — so that exactly one even theta-constant vanishes and thus $\Pi=0$ at codimension-one hyperelliptic locus in the moduli space at $g=3$. The deviation from the hyperelliptic locus is measured by $\sqrt{\Pi}$ which has modular weight 9, and therefore the relations between modular forms of lower weights (including those of weight 8, which are relevant for NSR measures) can be exhaustively studied in hyperelliptic terms, i.e. pure algebraically. To be more precise, if two forms of weight ≤ 8 coincide at hyperelliptic locus at genus 3, they coincide everywhere. At higher genera $g>3$ the codimension of hyperelliptic locus in the moduli space is higher: $(3g-3)-(2g-1)=g-2$. Of course, $\Pi=0$ at all these loci, but additional $g-3$ relations occur which should also be taken into account, and also Shottky relations should be added if one seeks for a description in terms of modular forms.

On hyperelliptic locus the modular transformations act by permutations of ramification points, and modular forms are just symmetric polynomials of a_i, multiplied by appropriate power of $\det\sigma$. This makes hyperelliptic parametrization extremely convenient for study of relations between modular forms, at least for low genera and weights.

				$\tau \to \tau+1$	$\tau \to -1/\tau$
θ_{00}^4	$=$	$b+c=$	a	b	$-a$
θ_{01}^4	$=$		b	a	$-c$
θ_{10}^4	$=$		c	$-c$	$-b$

Table 2: Modular transformations of genus-one theta-constants.

2.4 Relations between modular forms at particular genera

2.4.1 Genus one

Three theta-constants are related by Riemann identity

$$\theta_{00}^4 = \theta_{01}^4 + \theta_{10}^4 \equiv b + c \tag{2.24}$$

The space of modular forms at genus one is generated by two Eisenstein series:

$$E_4 = \sum_{m,n}{}' \frac{1}{(m+n\tau)^4} \sim \xi_4 = \sum_{e=1}^{3} \theta_e^8 = (b+c)^2 + b^2 + c^2 = 2\left(b^2 + bc + c^2\right) \tag{2.25}$$

and

$$\begin{aligned} E_6 &= \sum_{m,n}{}' \frac{1}{(m+n\tau)^6} \sim \left(\theta\begin{bmatrix}0\\1\end{bmatrix}^4 - \theta\begin{bmatrix}1\\0\end{bmatrix}^4\right)\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4 + \theta\begin{bmatrix}0\\1\end{bmatrix}^4\right)\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4 + \theta\begin{bmatrix}1\\0\end{bmatrix}^4\right) \\ &= (b-c)(2b+c)(b+2c) \end{aligned} \tag{2.26}$$

They are related to Dedekind function $\eta = e^{i\pi\tau/12}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\right)$ by

$$\eta^{24} = \Pi^8 = (\theta_{00}\theta_{01}\theta_{10})^8 = (bc(b+c))^2 = \frac{1}{1728}(E_4^3 - E_6^2) \tag{2.27}$$

For any of the three even theta-characteristic e we have:

$$2\theta_e^{16} - \theta_e^8 \sum_{e'}^{3} \theta_{e'}^8 = 2\langle e,*\rangle\theta_e^4 \prod_{e'}^{3} \theta_{e'}^4 = 2\langle e,*\rangle\theta_e^4\eta^{12} = 2\theta_e^4\Pi_*^4 \tag{2.28}$$

i.e.

$$\begin{aligned} 2(b+c)^4 - (b+c)^2 \cdot 2(b^2+bc+c^2) &= 2(b+c)\cdot bc(b+c) \\ 2b^4 - b^2 \cdot 2(b^2+bc+c^2) &= -2b \cdot bc(b+c) \\ 2c^4 - c^2 \cdot 2(b^2+bc+c^2) &= -2c \cdot bc(b+c) \end{aligned}$$

Thus for $g = 1$ the two vanishing-relations (2.13) and (2.19) are actually the same. Note that we absorbed the sign-factor $\langle e,*\rangle$ into the definition of Π_*^4.

Under modular transformations the theta-constants transform as shown in table 2. For

p	α_p	β_p	w_p
0	1	0	1
1	0	1	2
2	-2	3	4
3	-6	7	8
4	-14	15	16
...			
p	$-2(2^{p-1}-1)$	2^p-1	2^p

Table 3: Coefficients of $\xi_8^{(p)}$ linear decompositions (2.30) in two different basises at $g=1$.

$g=1$ all our forms of weights 4 and 8 are expressed through θ_e^8, and $\xi_4 = \sum_e \theta_e^8$:

$$\begin{aligned}
\xi_2[e] &\equiv \sum_{e'}^{3} \langle e,e'\rangle \theta_{e'}^4 = 2\theta_e^4, \\
\xi_{2,2} &\equiv \sum_{e,e'}^{3} \theta_e^4 \langle e,e'\rangle \theta_{e'}^4 = 2\sum_{e}^{3} \theta_e^8 = 2\xi_4, \\
\xi_6[e] &= \sum_{e'}^{3} \langle e,e'\rangle \theta_{e'}^{12} = -\theta_e^{12} + \frac{3}{2}\theta_e^4 \sum_{e'}^{3} \theta_{e'}^8 \overset{(2.28)}{=} \theta_e^4 \sum_{e'}^{3} \theta_{e'}^8 - \Pi_*^4 = \xi_4\theta_e^4 - \Pi_*^4, \\
\xi_{2,6} &\equiv \sum_{e,e'}^{3} \theta_e^4 \langle e,e'\rangle \theta_{e'}^{12} = 2\sum_{e}^{3} \theta_e^{16} = 2\xi_8 \overset{(2.19)}{=} \xi_4^2 = \left(\sum_{e}^{3} \theta_e^8\right)^2
\end{aligned} \tag{2.29}$$

For the set of the CDG-Grushevsky forms (2.9) and (2.10) we have:

$$\xi_8^{(p)}[e] = \alpha_p\, \theta_e^{16} + \beta_p\, \theta_e^8 \sum_{e'}^{3} \theta_{e'}^8 = \alpha_p\, \xi_8^{(0)}[e] + \beta_p\, \xi_8^{(1)}[e] \overset{(2.28)}{=} \frac{w_p}{2}\, \theta_e^8 \xi_4 + \alpha_p\, \theta_e^4 \Pi_*^4, \tag{2.30}$$

where $w_p = \alpha_p + 2\beta_p$. It follows that

$$\xi_8^{(p)} \equiv \sum_{e}^{3} \xi_8^{(p)}[e] = \frac{w_p}{2}\xi_4^2 = 2^{p-1}\xi_4^2 \tag{2.31}$$

Numerical coefficients α_p, β_p and w_p are easily evaluated, if theta-constants are expressed through b and c, see table 3.

In particular, it follows that $\xi_8^{(2)}[e] = 2\theta_e^4 \xi_6[e]$.

In hyperelliptic parametrization

$$\theta_{00}^4 = a_{12}a_{34}, \quad \theta_{01}^4 = a_{13}a_{24}, \quad \theta_{10}^4 = a_{41}a_{23} \tag{2.32}$$

and formulas look a little more involved than in terms of b and c, for example:

$$\xi_4 = \sum_e \theta_e^8 = a_{12}^2 a_{34}^2 + a_{13}^2 a_{24}^2 + a_{14}^2 a_{23}^2 = -6s_4 + 6s_3 s_1 + \frac{7}{2}s_2^2 - 4s_2 s_1^2 + \frac{1}{2}s_1^4, \tag{2.33}$$

S	$S\cup U$	$S\cap U$	$S\circ U$	θ_e^4
$\emptyset$	34	$\emptyset$	34	$\sim +\frac{1}{a_{31}a_{32}a_{41}a_{42}} \sim +a_{12}a_{34}$
13	134	3	14	$\sim -\frac{1}{a_{12}a_{13}a_{42}a_{43}} \sim -a_{14}a_{23}$
14	134	4	13	$\sim -\frac{1}{a_{12}a_{14}a_{32}a_{34}} \sim +a_{13}a_{24}$
23	234	3	24	$\sim -\frac{1}{a_{21}a_{23}a_{41}a_{43}} \sim +a_{13}a_{24}$
24	234	4	23	$\sim -\frac{1}{a_{21}a_{24}a_{31}a_{34}} \sim -a_{14}a_{23}$
1234	1234	34	12	$\sim +\frac{1}{a_{13}a_{14}a_{23}a_{24}} \sim +a_{12}a_{34}$
12	1234	$\emptyset$	1234	0
34	34	34	$\emptyset$	0

Table 4: Different ingredients of Thomae formula (2.22) at genus one.

where $s_m = \sum_{k=1}^4 a_i^k$. Also,

$$\begin{aligned}
\xi_8 &= \sum_e \theta_e^{16} = a_{12}^4a_{34}^4 + a_{13}^4a_{24}^4 + a_{14}^4a_{23}^4 = 2\xi_4^2 \\
\mathcal{R}_* &= \sum_e \langle e,*\rangle\theta_e^4 \sim a_{12}a_{34} - a_{13}a_{24} - a_{41}a_{23} = 0 \\
\xi_{2,2} &= a_{12}a_{34}(a_{12}a_{34} + a_{13}a_{24} + a_{41}a_{23}) + a_{13}a_{24}(a_{12}a_{34} + a_{13}a_{24} - a_{41}a_{23}) \\
&\quad + a_{41}a_{23}(a_{12}a_{34} - a_{13}a_{24} + a_{41}a_{23})
\end{aligned}$$

and

$$\begin{aligned}
\xi_{2,6} &= a_{12}a_{34}\left(a_{12}^3a_{34}^3 + a_{13}^3a_{24}^3 + a_{41}^3a_{23}^3\right) + a_{13}a_{24}\left(a_{12}^3a_{34}^3 + a_{13}^3a_{24}^3 - a_{41}^3a_{23}^3\right) \\
&\quad + a_{41}a_{23}\left(a_{12}^3a_{34}^3 - a_{13}^3a_{24}^3 + a_{41}^3a_{23}^3\right)
\end{aligned}$$

Still, all the relations, including (2.30), can be easily derived in this parametrization, and such derivations are straightforwardly generalized to $g = 2, 3$. The more economic b, c parametrization is also generalizable (it is related to expressions through theta-constants of doubled argument, $\theta(2T)$, which was actually used in [34]), but this is a slightly more involved technique, unnecessary for our presentation.

Formula (2.22) for $g = 1$ is represented by table 4. It is assumed here that $U = \{a_3, a_4\}$: this is the choice which reproduces (2.32). In the last two lines $\#(S\circ U) \neq g+1 = 2$, such sets S correspond to the odd characteristic with vanishing theta-constant.

2.4.2 Genus two

Of six (as many as there are odd characteristics *) Riemann identities (2.13) there are five linearly independent, and they express 10 a priori different θ_e^4 through 5 linearly independent ones. In addition there is one non-linear relation:

$$\chi_8 = 4\xi_8 - \xi_4^2 = 0, \qquad \text{i.e.} \quad \xi_8^{(0)} \equiv \xi_8 = \frac{1}{4}\xi_4^2, \qquad \xi_8^{(1)} = \xi_4^2 \tag{2.34}$$

Further,

$$\begin{aligned}
\xi_{2,2} &= 4\xi_4, \\
\xi_{2,6} &= 4\xi_8 = \xi_4^2
\end{aligned} \tag{2.35}$$

p	α_p	β_p	γ_p	w_p
0	1	0	0	1
1	0	1	0	4
2	0	0	1	16
3	8	-14	7	64
4	56	-90	35	256
$\dots$				
p	$\frac{8(2^{p-1}-1)(2^{p-2}-1)}{3}$	$-2(2^p-1)(2^{p-2}-1)$	$\frac{(2^p-1)(2^{p-1}-1)}{3}$	4^p

Table 5: Coefficients of $\xi_8^{(p)}$ linear decompositions (2.37) and (2.38) at $g=2$.

S	14 2356	16 2345	46 1235	23 1456	25 1346	35 1246
e(S)	$\begin{bmatrix} 0\ 1 \\ 0\ 1 \end{bmatrix}$	$\begin{bmatrix} 1\ 0 \\ 1\ 1 \end{bmatrix}$	$\begin{bmatrix} 1\ 1 \\ 1\ 0 \end{bmatrix}$	$\begin{bmatrix} 0\ 1 \\ 1\ 1 \end{bmatrix}$	$\begin{bmatrix} 1\ 1 \\ 0\ 1 \end{bmatrix}$	$\begin{bmatrix} 1\ 0 \\ 1\ 0 \end{bmatrix}$
θ_e	0	0	0	0	0	0

Table 6: Ingredients of of Thomae formula (2.22) at genus two for odd theta-characteristics.

and

$$\xi_8^{(2)}[e] = 4\theta_e^4\xi_6[e], \qquad \xi_8^{(2)} = \sum_e^{10} \xi_8^{(2)}[e] = 4\xi_{2,6} = 4\xi_4^2 \tag{2.36}$$

$$\begin{aligned} \xi_8^{(p)}[e] &= \alpha_p\,\theta_e^{16} + \beta_p\,\theta_e^8 \sum_{e'}^{3} \theta_{e'}^8 + \gamma_p\,\theta_e^4 \sum_{e',e''}^{3} \theta_{e'}^4\theta_{e''}^4\theta_{e+e'+e''}^4 \\ &= \alpha_p\,\xi_8^{(0)}[e] + \beta_p\,\xi_8^{(1)}[e] + \gamma_p\,\xi_8^{(2)}[e] \end{aligned} \tag{2.37}$$

It follows that

$$\xi_8^{(p)} \equiv \sum_e^{3} \xi_8^{(p)}[e] = \left(\frac{1}{4}\alpha_p + \beta_p + 4\gamma_p\right)\xi_4^2 = \frac{1}{4}\,w_p\,\xi_4^2 \tag{2.38}$$

where $w_p = \alpha_p + 4\beta_p + 16\gamma_p$. Numerical coefficients α_p, β_p and γ_p are easily evaluated if theta-constants are expressed in hyperelliptic parametrization, where they become simple algebraic relations, see table 5. The simplest way to prove this kind of identities is to use hyperelliptic parametrization, where they become simple algebraic relations. In the basis selected in [33] — it corresponds to taking $U = \{a_2, a_3, a_5\}$ in (2.22)[1] — we get expressions, collected in tables 6 and 7.

[1]However, association of theta-characteristics — the map $S \to e(S)$ — in [33] does not look consistent with the rule (2.23), and we choose another one in the second line of the table.

S	$\emptyset$ 123456	24 1356	13 2456	56 1234	26 1345	45 1236	15 2346	36 1245	34 1256	12 3456
$e(S)$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$
θ_e^4	$-a_{146}a_{235}$	$a_{126}a_{345}$	$a_{125}a_{346}$	$-a_{145}a_{236}$	$a_{124}a_{356}$	$-a_{156}a_{234}$	$a_{123}a_{456}$	$-a_{134}a_{256}$	$-a_{136}a_{245}$	$-a_{135}a_{246}$

Table 7: Ingredients of of Thomae formula (2.22) at genus two for even theta-characteristics.

Note that there is no direct counterpart of the relation (2.28) already for $g = 2$: the form $\chi_8 = 4\xi_8 - \xi_4^2$ is not a linear combination of Riemann identities (2.13). Moreover, one can easily check that it does not automatically vanish for arbitrary set of 5 linearly-independent θ_e^4: from genus two $\chi_8 = 0$ is an *additional* relation between theta-constants, algebraically (not only linear) independent of Riemann identities.

2.4.3 Genus three

The number N_* of Riemann identities is now 28, of which $\frac{4^g-1}{3} = 21$ are linearly independent and there are $\frac{(2^g+1)(2^{g-1}+1)}{3} = 36 - 21 = 15$ linearly independent θ_e^4. Again, there are additional non-linear relations, including

$$\chi_8 = 8\xi_8 - \xi_4^2 = \; 8\sum_e^{36}\theta_e^{16} - \left(\sum_e^{36}\theta_e^8\right)^2 = 0 \tag{2.39}$$

Hyperelliptic locus has codimension one in moduli space and is defined by $\Pi = \prod_e^{36}\theta_e = 0$. Still, hyperelliptic parametrization can be used to prove formulas at genus 3 for modular functions of weights ≤ 8, because deviations from hyperellipticity are proportional to $\sqrt{\Pi}$ which has weight 9.

2.4.4 Genus four

As shown in [63], and widely used since [3, 15, 17], $\chi_8 = 0$ exactly at the moduli space, embedded as codimension-one subspace in the Siegel upper semi-space. Hyperelliptic locus now has codimension $g - 2 = 4$, this is the place where $\Pi = 0$, but actually not just one, but 10 out of 136 even theta-constants vanish on it. Simple hyperelliptic calculations are still very useful here, but are not as conclusive as they are for $g < 4$.

3. Mumford measure for critical bosonic string [14, 15]

After a brief exposition of the theory of theta-constants – note that we do not need anything more than above simple statements — we are ready to switch to the string measures. As already mentioned in the Introduction, Belavin-Knizhnik theorem [3] expresses them through the holomorphic Mumford measure on the moduli space of complex curves, which has degree-2 poles at the boundaries: namely when one of the cycles (contractible or non-contractible) gets shrinked. The degree of the pole is controlled by the negative mass squared of a tachyon, present in the spectrum of bosonic string. Residues at the poles are given by two-point a function in the case of non-contractible cycle (when genus g curve degenerates into the one of $g-1$) and a product of two one-point functions in the case of contractible cycle (when the curve splits into two of genera g_1 and $g_2 = g - g_1$). In fact the values of pole degrees are enough to determine the measure and above properties can be used to read off expressions for one- and two-point functions. The most interesting object is the string measure on the *universal moduli space*, unifying all genera and all the correlators (scattering amplitudes) [70]. n-point correlators can also be promoted to stringy correlators by inclusion of Riemann surfaces with boundaries and/or non-oriented [71].

In fact all these generalizations are rather straightforward once the structure of string measures for particular genera is clarified[2] – and we list here original expressions from [14, 15]. For somewhat less explicit expressions for all genera see [4]–[8].

Genus one.

$$\frac{1}{(\mathrm{Im}\,\tau)^{14}}\left|\frac{d\tau}{\left(\prod_e^3\theta[e](\tau)\right)^8}\right|^2 \qquad \text{i.e.} \qquad d\mu = \frac{d\tau}{\Pi^8} \tag{3.1}$$

Genus two.

$$\frac{1}{\big(\det(\mathrm{Im}\,T)\big)^{13}}\left|\frac{dT_{11}dT_{12}dT_{22}}{\left(\prod_e^{10}\theta[e](\tau)\right)^2}\right|^2 \qquad \text{i.e.} \qquad d\mu = \frac{\prod_{i<j}^2 dT_{ij}}{\Pi^2} \tag{3.2}$$

Genus three.

$$\frac{1}{\big(\det(\mathrm{Im}\,T)\big)^{13}}\left|\frac{dT_{11}dT_{12}dT_{13}dT_{22}dT_{23}dT_{33}}{\left(\prod_e^{36}\theta[e](\tau)\right)^{1/2}}\right|^2 \qquad \text{i.e.} \qquad d\mu = \frac{\prod_{i<j}^3 dT_{ij}}{\sqrt{\Pi}} \tag{3.3}$$

Zero of the form in denominator is at the hyperelliptic locus. The square root singularity at this locus is fictitious: the period matrix in the vicinity of the locus is a *square* of the proper modulus [14, 15].

Genus four. This is the first time when the module space is smaller then Teichmuller one, it has complex codimension one and is defined by the zero of a single Shottky condition

$$\chi_8 = 0 \tag{3.4}$$

where χ_8 is the weight-8 modular form on Teichmuller space,

$$\chi_8(T) = 16\sum_e \theta[e]^{16} - \left(\sum_e \theta[e]^8\right)^2 \tag{3.5}$$

[2] The only subject which remains really puzzling concerns *arithmetic* properties of Mumford measure [72, 73]. Especially interesting is the relation between Polyakov and Migdal formalisms for string measures: the latter one is based on the use of equilateral triangulations, i.e. rational surfaces (Grothendieck's dessins d'enfant), which are not very well distributed inside the moduli space what makes equivalence of measures a kind of surprise, see [73] for details.

Bosonic string measure is

$$\frac{1}{\big(\det(\mathrm{Im}\,T)\big)^{13}}\left|\frac{\prod_{ij\le j}^{4} dT_{ij}}{\chi_8(T)}\right|^2 \tag{3.6}$$

This wonderful formula, suggested in [3] and [15] never attracted attention that it deserves and was not investigated as carefully as its lower-genera counterparts. Note that instead of the holomorphic delta-function of χ_8 in (3.6) one can put the sum of the NSR measures $\sum_e \Xi_8[e]$, which vanishes on the moduli space and is essentially the same as χ_8.

4. NSR measures

4.1 Superstring from NSR measures for fermionic string

Superstring possesses space-time supersymmetry in critical dimension $d = 10$. Two approaches are developed in order to describe it in the first quantization formalism, i.e. with the help of the two-dimensional actions on string world sheet. One approach (Green-Schwarz formalism [74]–[77]) is explicitly $d = 10$ supersymmetric, but the two-dimensional action is highly non-linear and possesses sophisticated κ-symmetry. Another, NSR approach [13, 56] is based on the theory of *fermionic string*, defined as possessing the world-sheet, i.e. $2d$ supersymmetry. On world sheets with non-trivial topologies one can impose a variety of boundary conditions on $2d$ fermions, associated with different spin-structures or, what is the same, the theta-characteristics. The corresponding holomorphic NSR measures $d\mu[e]$ on the moduli space of Riemann surfaces also depend on theta-characteristics. Fermionic string does not have $10d$ space-time supersymmetry, it has tachyon and divergencies, just as bosonic string. However, superstring Hilbert space is just a subspace in the Hilbert space of fermionic space, and the relevant GSO projection [56] is provided simply by a sum of any holomorphic conformal block over the spin-structures:

$$\langle A\rangle = \int \frac{1}{(\det(\mathrm{Im}\,T))^5}\left|\sum_e A[e]d\mu_[e]\right|^2 \tag{4.1}$$

where $A[e]$ is a combination of holomorphic Green functions, associated with the multipoint observable A.

In genus one the three NSR measures are well known [13]:

$$d\mu[e] = \frac{\langle e,*\rangle\theta_e^4 d\tau}{\eta^{12}}, \tag{4.2}$$

what means that they are expressed through Mumford measure $d\mu = \frac{d\tau}{\eta^{24}} = \frac{d\tau}{\Pi^8}$ from (3.1):

$$d\mu_e = \langle e,*\rangle\theta_e^4\eta^{12}d\mu = \theta_e^4\Pi_*^4 d\mu \tag{4.3}$$

where $*$ is the only odd theta-characteristic at $g = 1$. (Of course, for genus one the measure includes the 6-th power of $\mathrm{Im}\,\tau$ instead of the 5-th one in for $g > 1$.)

It is an old conjecture that the situation is similar for arbitrary genus:

$$d\mu[e] = \Xi_8[e]d\mu, \tag{4.4}$$

where $\Xi_8[e]$ is a semi-modular form of weight 8. This is a non-trivial hypothesis for $g \geq 4$, because there is no obvious reason why $d\mu[e]/d\mu$ should have any nice continuation to entire Siegel space, beyond the moduli space. Still, if this hypothesis is true, for any correlator in superstring theory we have a simple representation in terms of an integral over moduli space:

$$\langle A\rangle = \int \frac{|d\mu|^2}{(\det(\mathrm{Im}\, T))^5}\left|\sum_e A[e]\,\Xi_8[e]\right|^2 \tag{4.5}$$

Under these assumptions the only unknown is the set of forms $\Xi_8[e]$, which should satisfy two simple properties: factorization and the condition of vanishing cosmological constant,

$$\sum_e d\mu[e] = 0, \qquad \text{i.e.} \qquad \sum_e \Xi_8[e] = 0 \tag{4.6}$$

For genus 1 eq. (4.6) for (4.3) is an immediate corollary of the Riemann identity (2.13),

$$\sum_e \langle e,*\rangle \theta[e]^4 = 0 \tag{4.7}$$

It seemed a natural generalization of conjecture (4.4) to extend this property to all genera [41, 42]:

$$\Xi_8[e] \overset{?}{=} \langle e,*\rangle \theta_e^4 K_6^*, \tag{4.8}$$

especially because (2.14) would then automatically guarantee the vanishing of all $g \geq 1$ corrections to the $1, 2, 3$-point functions. Immediate drawback of this Riemann-identity hypothesis was explicit dependence on the odd characteristic $*$, which would un-acceptedly show up in non-vanishing 4-point function and in higher correlators. Worse than that, an appropriate form K_6^* does not seem to exist.

It was believed that the NSR measure can be *derived*, starting from explicitly $2d$-supersymmetric formalism for fermionic string, based on the clever definition of super-Riemann surfaces, by integrating over odd supermoduli. However, naive simplified approaches of this kind (attempting to trivialize the supermoduli bundle over the ordinary module space) failed, and accurate integration was performed only recently in [17]–[20] and only for $g = 2$. The outcome was a confirmation of hypothesis (4.4) and a clear denunciation of (4.8): it appeared that instead of continuing (4.7) from $g = 1$ to $g > 1$ one should rather substitute it by

$$\begin{aligned} \mathrm{g} = 1: \qquad \Xi_8[e] &= \sum_e \langle e,*\rangle \theta[e]^4 \Pi_*^4 \\ &\overset{(2.28)}{=} 2\sum_e \theta_e^{16} - \left(\sum_e \theta_e^8\right)^2 = \chi_8 \overset{(2.9)}{=} 2\xi_8^{(0)} - \xi_8^{(1)} \end{aligned} \tag{4.9}$$

and continue the r.h.s. (note that relation (2.28) does not survive at $g \geq 2$, so that continuations of its two sides deviate from each other). Such continuation was *derived* in [17]–[20] for $g = 2$, reformulated and generalized to $g = 3, 4$ in [33, 34, 36] and was put in the nice form, conjecturally reasonable for arbitrary g in [35]. Since CPG-Grushevsky conjecture for $g \geq 3$ expresses $d\mu[e]$ through $\xi_8^{(p)}$ with $p \geq 3$, it does not contain an explicit θ_e^4 factor, what makes puzzling the story about the $1, 2, 3$-point functions.

4.2 Anzatz for the NSR measures [17, 34, 36]

The natural generalization of the r.h.s. of (4.9) is

$$\text{any g}: \qquad \Xi_8[e] = \sum_{p=0}^{g} h_p \, \xi_8^{(p)}[e], \tag{4.10}$$

where CDG-Grushevsky forms at the r.h.s. are defined in (2.9) and (2.10) and coefficients h_p are constrained by requirements of factorization and vanishing of the cosmological constant.

The latter one implies that

$$\sum_{e}^{N_e} \Xi_8[e] = \sum_{p=0}^{g} h_p \, \xi_8^{(p)} = 0 \tag{4.11}$$

Since the l.h.s. is a modular form of weight 8, it should be proportional to $\xi_4^2 \overset{(2.19)}{=} 2^g \xi_8$ and the same is true for all the terms in the sum:

$$\xi_8^{(p)} = \frac{1}{2} W_p \, \xi_4^2 \tag{4.12}$$

Thus the requirement (4.11) simply states that

$$\sum_{p=0}^{g} h_p W_p = 0 \tag{4.13}$$

Coefficients W_p can be evaluated by different methods, but the simplest one is to go to the high-codimension subset at the boundary of moduli space, when the curve degenerates into a set of tori and period matrix T becomes diagonal $T = \mathrm{diag}(\tau_1, \ldots, \tau_g)$. Then $\xi_4(T) \to \prod_{i=1}^{g} \xi_4(\tau_i) = \xi_4^{\otimes g}$ and

$$\xi_8^{(p)}(T) \longrightarrow \prod_{i=1}^{g} \xi_8^{(p)}(\tau_i) \overset{(2.31)}{=} \left(\frac{w_p}{2}\right)^g \prod_{i=1}^{g} \xi_4^2(\tau_i) \tag{4.14}$$

so that

$$W_p = 2 \left(\frac{w_p}{2}\right)^g \overset{\text{table 3}}{=} 2^{g(p-1)+1} \tag{4.15}$$

Of course, (4.13) is an important but non-restrictive constraint on the coefficients h_p. All the h_p are determined if the same reduction to genus one is made for the individual $\Xi_8[e]$: On one side,

$$\Xi_8[e](T) \to \prod_{i=1}^{g} \Xi_8[e_i](\tau_i) \overset{(4.3)}{=} \prod_{i=1}^{g} \left\{ \theta_{e_i}^4 \Pi_*^4(\tau_i) \right\} \tag{4.16}$$

g	h_0	h_1	h_2	h_3	h_4	h_5	$\ldots$
1	1	$-\frac{1}{2}$					
2	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{12}$				
3	$\frac{8}{21}$	$-\frac{1}{3}$	$\frac{1}{12}$	$-\frac{1}{168}$			
4	$\frac{64}{315}$	$-\frac{4}{21}$	$\frac{1}{18}$	$-\frac{1}{168}$	$\frac{1}{5040}$		
5	$\frac{1024}{9765}$	$-\frac{32}{315}$	$\frac{2}{63}$	$-\frac{1}{252}$	$\frac{1}{5040}$	$-\frac{1}{312480}$	
$\ldots$							

Table 8: The values (4.19) of coefficients h_p in (4.10) for the first low genera g.

on another side

$$\begin{aligned}\Xi_8[e](T) &\overset{(4.11)}{=} \sum_{p=0}^{g} h_p\, \xi_8^{(p)}[e] \longrightarrow \sum_{p=0}^{g} h_p \left\{\prod_{i=1}^{g} \xi_8^{(p)}[e_i](\tau_i)\right\} \\ &\overset{(2.30)}{=} \sum_{p=0}^{g} h_p \left\{\prod_{i=1}^{g} \left(\frac{w_p}{2}\theta_{e_i}^8 \xi_4 + \alpha_p\, \theta_{e_i}^4 \Pi_*^4\right)(\tau_i)\right\} \end{aligned} \tag{4.17}$$

Comparing the two expressions we obtain a set of $g+1$ linear equations for $g+1$ coefficients h_p:

$$\sum_{p=0}^{g} h_p w_p^k (2\alpha_p)^{g-k} = 2^g \delta_{k,0} \qquad \text{or} \qquad \sum_{p=0}^{g} \tilde{h}_p\, \lambda_p^k = 2^g \delta_{k,0} \tag{4.18}$$

with $k = 0, \ldots, g$, $\tilde{h}_p = (2\alpha_p)^g \tilde{h}_p$ and $\lambda_p = w_p/2\alpha_p$, so that h_p is the ratio of Van-der-Monde determinants:

$$\tilde{h}_p = 2^g \frac{\Delta_p(\lambda)}{\Delta(\lambda)} = 2^g \prod_{i \neq p} \frac{\lambda_i}{\lambda_i - \lambda_p} \qquad \text{and} \qquad h_p = \prod_{i \neq p} \frac{w_i}{w_i \alpha_p - w_p \alpha_i} \tag{4.19}$$

It is easy to check, that the vanishing relations (4.13) and thus (4.11) are true with these values of h_p.

In Grushevsky's basis [35] the coefficients are much nicer, moreover, they are actually independent of g. Indeed, substituting $\xi_8^{(p)}$ in the form (2.11) and h_p from the table 8 into (4.10) we obtain table 9 and finally

$$d\mu[e] = \Xi_8[e] d\mu, \qquad \Xi_8[e] = \frac{1}{2^g} \sum_{p=0}^{g} \frac{(-)^p}{\prod_{i=1}^{p}(2^i - 1)} G_8^{(p)}[e] \tag{4.20}$$

(the coefficient in the term with $p = 0$ is unity, by the usual rule $\prod_1^0 = 1$, like $0! = 1$). Note that in [35] the normalization of $G_8^{(p)}$ was chosen differently, therefore the coefficients in (4.20) are also different.

$g=1$	$\Xi_8[e] = \frac{1}{2}\Big(G_8^0[e] \ - G_8^{(1)}[e]\Big)$
$g=2$	$\Xi_8[e] = \frac{1}{4}\Big(G_8^0[e] \ - \ G_8^{(1)}[e] \ + \frac{1}{3}G_8^{(2)}[e]\Big)$
$g=3$	$\Xi_8[e] = \frac{1}{8}\Big(G_8^0[e] \ - \ G_8^{(1)}[e] \ + \ \frac{1}{3}G_8^{(2)}[e] \ - \ \frac{1}{21}G_8^{(3)}[e]\Big)$
$g=4$	$\Xi_8[e] = \frac{1}{16}\Big(G_8^0[e] - \ G_8^{(1)}[e] \ + \ \frac{1}{3}G_8^{(2)}[e] \ - \ \frac{1}{21}G_8^{(3)}[e] \ + \frac{1}{315}G_8^{(4)}[e]\Big)$
$g=5$	$\Xi_8[e] = \frac{1}{32}\Big(G_8^0[e] - \ G_8^{(1)}[e] \ + \ \frac{1}{3}G_8^{(2)}[e] \ - \ \frac{1}{21}G_8^{(3)}[e] \ + \ \frac{1}{315}G_8^{(4)}[e] \ - \frac{1}{9765}G_8^{(5)}[e]\Big)$
...	

Table 9: The NSR-measure weight-8 form $\Xi_8[e]$ in Grushevsky's basis.

4.3 More degeneration examples

In addition to (4.14) one can consider reductions to lower-codimension components of the boundary, where, for example, the curve degenerates into two of genera g_1 and g_2 with $g_1 + g_2 = g$. This is an important check, but the result actually follows from above much simpler consideration.

For example, the genus-three

$$\Xi_8 \overset{(4.10)}{=} \frac{8}{21}\xi_8^{(0)} - \frac{1}{3}\xi_8^{(1)} + \frac{1}{12}\xi_8^{(2)} - \frac{1}{168}\xi_8^{(3)} \tag{4.21}$$

decomposes into genus-one and genus-two quantities

$$\begin{aligned}
\Xi_8 \longrightarrow \Xi_8\begin{pmatrix} \tau & 0 & 0 \\ 0 & T_{11} & T_{12} \\ 0 & T_{12} & T_{22}\end{pmatrix} &= \frac{8}{21}\xi_8^{(0)}(\tau)\otimes\xi_8^{(0)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} - \frac{1}{3}\xi_8^{(1)}(\tau)\otimes\xi_8^{(1)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} \\
&\quad + \frac{1}{12}\xi_8^{(2)}(\tau)\otimes\xi_8^{(2)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} - \frac{1}{168}\xi_8^{(3)}(\tau)\otimes\xi_8^{(3)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} \\
&\overset{(4.23)}{=} \left(\xi_8^{(0)} - \frac{1}{2}\xi_8^{(1)}\right)(\tau)\otimes\left(\frac{2}{3}\xi_8^{(0)} - \frac{1}{2}\xi_8^{(1)} + \frac{1}{12}\xi_8^{(2)}\right)\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} \\
&\overset{(4.10)}{=} \Xi_8(\tau)\otimes\Xi_8\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix}
\end{aligned} \tag{4.22}$$

where we substituted the genus-one and genus-two relations:

$$\begin{aligned}
\xi_8^{(2)}(\tau) &\overset{\text{table 3}}{=} -2\xi_8^{(0)}(\tau) + 3\xi_8^{(1)}(\tau), \\
\xi_8^{(3)}(\tau) &\overset{\text{table 3}}{=} -6\xi_8^{(0)}(\tau) + 7\xi_8^{(1)}(\tau)
\end{aligned} \tag{4.23}$$

and

$$\xi_8^{(3)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} \overset{\text{table 5}}{=} 8\xi_8^{(0)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} - 14\xi_8^{(1)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} + 7\xi_8^{(2)}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22}\end{pmatrix} \tag{4.24}$$

We omit characteristics labels in this section to simplify the formulas.

$\xi^{(0)} \otimes \xi^{(0)}$	$H_0 + 8^2 H_3 + 56^2 H_4 = h_0^2$	$\frac{64}{315} - \frac{8^2}{168} + \frac{56^2}{5040} = \frac{4}{9}$
$\xi^{(1)} \otimes \xi^{(1)}$	$H_1 + 14^2 H_3 + 90^2 H_4 = h_1^2$	$-\frac{4}{21} - \frac{14^2}{168} + \frac{90^2}{5040} = \frac{1}{4}$
$\xi^{(2)} \otimes \xi^{(2)}$	$H_2 + 7^2 H_3 + 35^2 H_4 = h_2^2$	$\frac{1}{18} - \frac{7^2}{168} + \frac{35^2}{5040} = \frac{1}{144}$
$\xi^{(0)} \otimes \xi^{(1)}$	$- 8 \cdot 14 H_3 - 56 \cdot 90 H_4 = h_0 h_1$	$\frac{112}{168} - \frac{56 \cdot 90}{5040} = -\frac{1}{3}$
$\xi^{(0)} \otimes \xi^{(2)}$	$8 \cdot 7 H_3 + 56 \cdot 35 H_4 = h_0 h_2$	$- \frac{56}{168} + \frac{56 \cdot 35}{5040} = \frac{1}{18}$
$\xi^{(1)} \otimes \xi^{(2)}$	$- 7 \cdot 14 H_3 - 90 \cdot 35 H_4 = h_1 h_2$	$\frac{98}{168} - \frac{90 \cdot 35}{5040} = -\frac{1}{24}$

Table 10: Coefficients in front of the different structures in eq. (4.26).

Similarly, to check the decomposition with $g = g_1 + g_2$,

$$\Xi_8 = \sum_{p=0}^{g} h_p \xi_8^{(p)} \longrightarrow \underline{\sum_{p=0}^{g} h_p \xi_8^{(p)} \otimes \xi_8^{(p)}} = \left(\sum_{p=0}^{g_1} h_p \xi_8^{(p)} \right) \otimes \left(\sum_{p=0}^{g_2} h_p \xi_8^{(p)} \right) = \Xi_8 \otimes \Xi_8 \quad (4.25)$$

one needs to know the analogues of (2.30) and (2.37) to substitute into the underlined expression. After that the next equality is just an algebraic identity for the coefficients h_p in the table 8. Remarkably, generalizations of (2.30) and (2.37) can be found for all genera by pure algebraic means: analyzing restrictions to hyperelliptic loci. Despite these loci have high codimension $g - 2$, all the coefficients are unambiguously fixed in these restrictions. Eqs. (2.30) and (2.37) themselves are actually enough to validate decompositions $g = m \cdot 1 + n \cdot 2$ with various m and n.

To show just one more example, the decomposition $4 \to 2 + 2$ implies that

$$\begin{aligned} H_0 \xi^{(0)} \otimes \xi^{(0)} + H_1 \xi^{(1)} \otimes \xi^{(1)} + H_2 \xi^{(2)} \otimes \xi^{(2)} + H_3 \xi^{(3)} \otimes \xi^{(3)} + H_4 \xi^{(4)} \otimes \xi^{(4)} = \\ = \left(h_0 \xi^{(0)} + h_1 \xi^{(1)} + h_2 \xi^{(2)} \right) \otimes \left(h_0 \xi^{(0)} + h_1 \xi^{(1)} + h_2 \xi^{(2)} \right) \end{aligned} \quad (4.26)$$

where H_p correspond to genus 4 (the forth line in table 8) while h_p — to genus 2 (the second line in table 8), — and genus-two modular forms $\xi_8^{(p)}[e]$ are related by (2.37):

$$\begin{aligned} \xi^{(3)} &= 8\xi^{(0)} - 14\xi^{(1)} + 7\xi^{(2)}, \\ \xi^{(4)} &= 56\xi^{(0)} - 90\xi^{(1)} + 35\xi^{(2)} \end{aligned} \quad (4.27)$$

Collecting the coefficients at different independent products of forms in (4.26), we obtain table 10. Equalities in the last column obtained by substitution of the coefficients from the table 8 are indeed true.

5. Conclusion

To conclude, we reviewed spectacular new development in perturbative superstring theory, caused by the ground-breaking papers [17]–[25] of Eric D'Hoker and Duong Phong and their direct continuation in [31]–[37]. The main reason why these formulas have not been discovered in the first attack on NSR measures in 1980's seems related to three prejudices.

First, starting from [41], the vanishing of cosmological constant was attributed to Riemann identities, while the simple relation (2.28) at genus one allowed two kinds of generalizations: to (2.13) and to (2.19). It turned out that the second choice is more appropriate.

Second, NSR measure $d\mu_e$ was believed to be proportional to θ_e^4, so that expressions for to 1,2,3,4-point functions would not contain θ_e in denominators. Remarkably, this prejudice was still alive in [17] and was finally broken only in [34], though it was actually based on the misleading overestimate of the role of the Riemann identities (since they had a generalization (2.14), the vanishing of 1,2,3-point functions would automatically come together with that of the 0-function — if Riemann identities were the right thing to rely upon).

Third, naive integration over odd supermoduli was associated with a correlator of the superghost β,γ-fields [53], which produced a non-trivial theta-function in *denominator* and summation over spin structures (theta-characteristics) looked hopeless. An artistic choice of odd moduli was then required in order to eliminate this theta-function and perform the summation. Exact treatment of odd moduli in [17]–[25] confirmed that the measure $d\mu_e$ is simple and has nothing non-trivial in denominator (at least for genus two) and this opened the way for a new stage of guess-work, based on the search of the modular forms with given properties.

Today all these problems seem to be largely resolved, *the outcome — eqs. (4.10) and (4.20)* — is nearly obvious (once you know it) and it *deserves to be widely known.* Our main goal in this text was to give as simple presentation of the subject as possible, avoiding unnecessary details about supermoduli integration and modular-forms theory, relying instead only on widespread knowledge of elementary string theory. To avoid overloading the text we did not include consideration of non-renormalization theorems for 1,2,3-point functions [38], in particular, the resolution of the θ_e^4 "paradox", and the most interesting expressions for 4-point functions (found and proved in above-cited references). Already at the level of 4-point functions the NSR string with GSO projection can be compared to Green-Schwarz superstring [74]–[76], where equally impressive progress is also achieved in recent years due to the works of Nathan Berkovits [77] — and this is a separate issue of great importance to be addressed elsewhere.

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